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Asymptotic behaviors in stochastic heat equations with periodic coefficients

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Abstract

This note is on the attempt to study the asymptotic behaviors in stochastic partial differential equation via Kipnis-Varadhan's theory on functional central limit theorem. In this note we considered a stochastic heat equation with periodic coefficients, which is closely related to the dynamical sine-Gordon equation. We conclude that under time scale $t^{-\frac{1}{2}}$, the law of the solution will converge to a centered Gaussian distribution as $t\rightarrow\infty$, and the fluctuation in x will vanish.

1 Stochastic heat equations

Given a Hilbert space H, the cylindrical Brownian motion W_{t} on H is defined formally by the series

$$W_{t}=\sum_{j=0}^{\infty}B_{t}^{j}e_{j},\ t\geq 0, \tag{1.1}$$

where $\{e_{j}\}$ is a CONS of H and $\{B_{t}^{j}\}$ is an infinite sequence of independent standard 1-dimensional Brownian motions. Notice that (1.1) does not converge in H; indeed the expected value of the H-norm $E\|W_{t}\|^{2}=\infty$. Instead, it converges in another Hilbert space H' containing H with a Hilbert-Schmidt embedding.

Suppose that $V_{x}(\cdot)=V(x, \cdot)$ is a family of C^{1} functions on $\mathbb{R}$ indexed by $x\in[0,1]$, and $V_{x}'(u)=\frac{d}{du}V_{x}(u)$ for $u\in \mathbb{R}$. We deal with the following 1-dimensional stochastic PDE with a Neumann boundary condition

$$\begin{cases}\partial_{t}u(t,x)=\dfrac{1}{2}\partial_{x}^{2}u(t,x)-V_{x}'(u(t,x))+\dot{W}(t,x), & t>0, x\in(0,1),\\ \partial_{x}u(t,0)=\partial_{x}u(t,1)=0, & t>0,\\ u(0,x)=v(x), & x\in[0,1],\end{cases} \tag{1.2}$$

where W is a cylindrical Brownian motion on $L^{2}[0,1]$ and $\dot{W}(t,x)$ is formally its derivative in x. Precisely, by the solution to (1.2) we mean a process $u(t)\in L^{2}[0,1]$ such that for all $\varphi \in C^2[0,1]$, $\varphi'(0) = \varphi'(1) = 0$,

$$\langle u(t), \varphi \rangle = \langle v, \varphi \rangle + \int_0^t V^\varphi(u(r))dr + \langle W_t, \varphi \rangle, \tag{1.3}$$

where $\langle W_t, \varphi \rangle$ is a Brownian motion and V^φ is a functional on $C[0,1]$ defined as

$$V^\varphi(v) \triangleq \frac{1}{2}\int_0^1 v(x)\varphi''(x)dx - \int_0^1 V_x'(v(x))\varphi(x)dx.$$

The stochastic PDE (1.2) is originally defined in [2] for the purpose of describing the motion of a flexible Brownian string in some potential field. In this note we need the following assumptions on V_x:
(1) $\forall u \in \mathbb{R}$, $V_x(u)$ is Borel-measurable in x;
(2) $\sup_{x\in[0,1],u\in\mathbb{R}} \{|V_x(u)| + |V_x'(u)|\} < \infty$;
(3) $\forall x \in [0,1]$, V_x' is global Lipschitz continuous with the same Lipschitz constant.
(4) $\forall x \in [0,1]$, V_x is periodic in u: $V_x(u) = V_x(u+1)$.
Under condition (1)-(3), the solution $u(t)$ uniquely exists in $C[0,1]$ and forms a continuous Markov process. Furthermore, if $\{w_x\}_{x\in[0,1]}$ is a 1-dimensional Brownian motion whose initial distribution is the Lebesgue measure on $\mathbb{R}$, then the reversible measure of $u(t)$ is an infinite measure on $C[0,1]$ given by

$$\mu(dv) = \exp\left\{-2\int_0^1 V_x(v(x))dx\right\}\mu_w(dv), \tag{1.4}$$

where μ_w stands for the measure induced by w_x (see in [2]).

This model is closely related to the following dynamical sine-Gordon model

$$\partial_t u = \frac{1}{2}\Delta u + c\sin(\beta u + \theta) + \xi, \tag{1.5}$$

where c, β and θ are real constants and ξ denotes the space-time white noise. As introduced in [3], (1.5) is the natural dynamic associated to the usual quantum sine-Gordon model. From a physical perspective, (1.5) describes globally neutral gas of interacting charges at different temperature β. When the spacial dimension is 2 or more, to construct the solution to (1.5) we need Hairer's theory of regularity structures (see in [3]). Now we restrict our discussion to the 1-dimensional case. The aim of this note is to study the limit distribution of $u(t)/\sqrt{t}$. Our main results are listed below.

Theorem 1.1. *Under an initial probability distribution ν such that $\nu \ll \mu$,*

$$\lim_{t\to\infty} E_\nu \left| \mathbb{E}\left[f\left(\frac{u(t)}{\sqrt{t}}\right) \middle| \mathcal{F}_0 \right] - \int_{\mathbb{R}} f(\mathbf{1}\cdot y)N_{\sigma^2}(dy) \right| = 0 \tag{1.6}$$

holds for all $f \in C_b(C[0,1])$, where σ is a constant introduced later and N_{σ^2} stands for a 1-dimensional centered Gaussian distribution on $\mathbb{R}$ with variance σ^2.

Theorem 1.2. *Under initial distribution $\nu \ll \mu$, $\{\epsilon u(\epsilon^{-2}t), t \in [0,T]\}$ converges weakly to a Gaussian process $\{\sigma B_t \cdot \mathbf{1}, t \in [0,T]\}$ as $\epsilon \downarrow 0$, where $T > 0$ is fixed, B_t is a 1-dimensional Brownian motion on $[0,T]$ and σ is the same constant as in Theorem 1.1.*

2 CLT and invariance principle

A general theory of functional CLT for Markov processes is developed in [4], based on a martingale-decomposition of the targeted functional. This method is extended to non-reversible cases in many references, e.g. [6], [7], [8] and [10]. Combined with Itô's formula, it can be used to prove the central limit theorem for diffusion processes in $\mathbb{R}^d$ with periodic coefficients, as illustrated in [5, Chapter 9]. We use the same strategy to prove Theorem 1.1.

Consider an equivalence relation in $C[0,1]$ such that $v_1 \sim v_2$ if and only if $v_1 - v_2$ equals to some integer-valued constant function. Let $\dot{E} = C[0,1]/\sim$ and identify $\dot{v} \in \dot{E}$ with its representative $v \in C[0,1]$ such that $v(0) \in [0,1)$. A function f on $C[0,1]$ can be automatically regarded as a function on $\dot{E}$ if it satisfies that $f(v+\mathbf{1}) = f(v)$. Let $\dot{u}(t)$ be the process induced by $u(t)$ on $\dot{E}$. Notice that $\dot{u}(t)$ is well-defined because we have condition (4) on the periodicity of coefficients.

It is clear that $\dot{u}(t)$ inherits the Markov property and a finite reversible measure form $u(t)$. Precisely, suppose $\{w'_x\}_{x\in[0,1]}$ to be a 1-dimensional Brownian motion whose initial distribution is the Lebesgue measure on $[0,1)$, then

$$\pi(d\dot{v}) = \frac{1}{Z}\exp\left\{-2\int_0^1 V_x(\dot{v}(x))dx\right\}\pi_w(d\dot{v}) \tag{2.1}$$

is a probability measure and is reversible for $\dot{u}(t)$, where π_w stands for the measure of w'_x and Z is a normalization constant. Let $\mathcal{H}$ be the Hilbert space $L^2(\dot{E},\pi)$, with the inner product $\langle\cdot,\cdot\rangle_\pi$ and the norm $\|\cdot\|_\pi$. Denote by $\{\dot{\mathcal{P}}_t\}$ the Markov semigroup generated by $\dot{u}(t)$ on $\mathcal{H}$. Recall the results in [9] on the strong Feller property and irreducibility of $\{\dot{\mathcal{P}}_t\}$, we can conclude that π is the only one invariant measure, thus it is ergodic.

Let $\mathcal{E}_A(H)$ be the linear span of all real and imaginary parts of functions on H of the form $h \mapsto e^{i\langle l,h\rangle}$ where $l \in C^2[0,1]$ such that $l'(0) = l'(1) = 0$. Moreover, suppose $\mathcal{E}_A(\dot{E})$ to be the collection of functions in $\mathcal{E}_A(H)$ such that $f(v) = f(v+\mathbf{1})$ for all $v \in E$. For $f \in \mathcal{E}_A(\dot{E})$, define

$$\dot{\mathcal{K}}_0 f(\dot{v}) = \frac{1}{2}\langle\partial_x^2 Df(\dot{v}), v\rangle + \frac{1}{2}Tr\left[D^2 f(\dot{v})\right] - \langle Df(\dot{v}), V'_\cdot(v(\cdot))\rangle, \tag{2.2}$$

where D denotes the Fréchet derivative. The integration-by-part formula for Wiener measure suggests that

$$E_\pi\|Df\|^2 = 2\langle f, -\dot{\mathcal{K}}_0 f\rangle_\pi, \tag{2.3}$$

thus $\dot{\mathcal{K}}_0$ is dissipative on $\mathcal{H}$. Denote its closure by $(\mathcal{D}(\dot{\mathcal{K}}),\dot{\mathcal{K}})$. Along a similar strategy used in [1], we can conclude that $\dot{\mathcal{K}}$ generates $\{\dot{\mathcal{P}}_t\}$ on $\mathcal{H}$. For $f \in \mathcal{E}_A(\dot{E})$ let

$$\|f\|_1^2 = \langle -\dot{\mathcal{K}}f, f\rangle_\pi = \frac{1}{2}E_\pi\|Df\|^2.$$

Let $\mathcal{H}_1$ be completion of $\mathcal{E}_A(\dot{E})$ under $\|\cdot\|_1$, which turns to be a Hilbert space if all f such that $\|f\|_1 = 0$ are identified with 0. On the other hand, let

$$\mathcal{I} = \left\{f \in \mathcal{H};\, \|f\|_{-1} \triangleq \sup_{g\in\mathcal{E}_A(\dot{E}),\|g\|_1=1}\langle f,g\rangle_\pi < \infty\right\}$$

Let $\mathcal{H}_{-1}$ be the completion of $\mathcal{I}_{-1}$ under $\|\cdot\|_{-1}$, which also becomes a Hilbert space if all f with $\|f\|_{-1} = 0$ are identified with 0. Denote by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_{-1}$ the inner products in $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ defined by polarization respectively.

Proposition 2.1. *For all $f \in \mathcal{D}(\dot{\mathcal{K}})$, the following equation holds π-a.s. and in $\mathcal{H}$.*

$$f(\dot{u}(t)) = f(\dot{u}(0)) + \int_0^t \dot{\mathcal{K}} f(\dot{u}(r))dr + \int_0^t \langle Df(\dot{u}(r)), dW_r\rangle. \tag{2.4}$$

Proof. When $f \in \mathcal{E}_A(\dot{E})$, (2.4) follows from the classical Itô's formula easily. For general f, since $\dot{\mathcal{K}}$ is the closure of $(\mathcal{E}_A(\dot{E}), \dot{\mathcal{K}}_0)$, we can pick $f_m \in \mathcal{E}_A(\dot{E})$ such that $f_m \to f$, $\dot{\mathcal{K}} f_m \to \dot{\mathcal{K}} f$ in $\mathcal{H}$. Then (2.3) suggests that $\|Df_m - Df\|$ also vanishes in $\mathcal{H}$ as $m \to \infty$. Therefore, (2.4) follows from the Itô isometry. □

Proof of Theorem 1.1. Pick $\varphi \in C^2[0,1]$ such that $\varphi'(0) = \varphi'(1) = 0$. Recall (1.3), it is not hard to verify that $V^\varphi \in \mathcal{H} \cap \mathcal{H}_{-1}$ and $\|V^\varphi\|_{-1} \le \frac{\sqrt{2}}{2}\|\psi\|$. For $\lambda > 0$ we consider the resolvent equation written as

$$\lambda f_\lambda^\varphi - \dot{\mathcal{K}} f_\lambda^\varphi = V^\varphi. \tag{2.5}$$

Taking inner product with f_λ^φ in (2.5), since $\dot{u}(t)$ is reversible under π we have

$$\sup_{\lambda>0} \|\dot{\mathcal{K}} f_\lambda^\varphi\|_{-1} = \sup_{\lambda>0} \|f_\lambda^\varphi\|_1 \le \|V^\varphi\|_{-1} < \infty. \tag{2.6}$$

Decompose the additive functional as $\int_0^t V^\varphi(\dot{u}(r))dr = M_\lambda^\varphi(t) + R_\lambda^\varphi(t)$, where M_λ^φ is the Dynkin's martingale and R_λ^φ is the residual term

$$M_\lambda^\varphi(t) = f_\lambda^\varphi(\dot{u}(t)) - f_\lambda^\varphi(\dot{u}(0)) - \int_0^t \dot{\mathcal{K}} f_\lambda^\psi(\dot{u}(r))dr,$$

$$R_\lambda^\varphi(t) = f_\lambda^\varphi(\dot{u}(0)) - f_\lambda^\varphi(\dot{u}(t)) + \lambda \int_0^t f_\lambda^\psi(\dot{u}(r))dr.$$

Applying (2.4) to f_λ^φ, combining it with this decomposition, we have

$$\langle u(t), \varphi\rangle = \langle u(0), \varphi\rangle + \int_0^t \langle Df_\lambda^\varphi(\dot{u}(r)) + \varphi, dW_r\rangle + R_\lambda^\varphi(t).$$

Condition (2.6) implies that (see in [5, Chapter 2]) there exists some $f^\varphi \in \mathcal{H}_1$ and an adapted process $R^\varphi(t)$ such that

$$\langle u(t), \varphi\rangle = \langle u(0), \varphi\rangle + \int_0^t \langle Df^\varphi(\dot{u}(r)) + \varphi, dW_r\rangle + R^\varphi(t).$$

Now the vanishment of $R^\varphi(t)$ (see in [5, Chapter 2]) and martingale CLT show that under initial distribution $\nu \ll \mu$,

$$\lim_{t\to\infty} E_\nu \left| \mathbb{E}\left[f\left(\frac{\langle u(t), \varphi\rangle}{\sqrt{t}}\right) \middle| \mathcal{F}_0 \right] - \int_{\mathbb{R}} f(y) N_{\sigma^2}(dy) \right| = 0 \tag{2.7}$$

for all $f \in C_b(\mathbb{R})$ and $\theta \in \mathbb{R}$, where $\sigma_\varphi^2 = E_\pi \|Df^\varphi + \varphi\|^2$.

Finally, to prove Theorem 1.1 we only need to pick $\varphi = e_j$ in (2.7) such that $\{e_j\}$ forms a CONS of $L^2[0,1]$ including the constant function $\mathbf{1}$ and sum them up. □

Proof of Theorem 1.2. Fix $T > 0$ and it is sufficient to verify the tightness of the laws of the processes $\epsilon u(\epsilon^{-2}\cdot)$ when $\epsilon \downarrow 0$. Let $S(t)$ be the semigroup generated by $\frac{1}{2}\partial_x^2$ on $L^2[0,1]$, then $u(t)$ satisfies that

$$u(t) = S(t)v + \int_0^t S(t-r)[-V'_\cdot(u(r,\cdot))]dr + \int_0^t S(t-r)dW_r$$

Denote the three terms in the right-hand side by $X(t)$, $Y(t)$ and $Z(t)$ respectively. Furthermore, let $X^\perp(t) \triangleq X(t) - \int_0^1 X(t,x)dx$ and define $Y^\perp$, $Z^\perp$ similarly. Then

$$\epsilon u(\epsilon^{-2}t) = \epsilon \int_0^1 u(\epsilon^{-2}t, x)dx + \epsilon X^\perp(\epsilon^{-2}t) + \epsilon Y^\perp(\epsilon^{-2}t) + \epsilon Z^\perp(\epsilon^{-2}t).$$

When $\epsilon \downarrow 0$, [5, Theorem 2.32] yields that the integral term is tight, while $\{\epsilon X^\perp(\epsilon^{-2}t), t \in [0,T]\}$ vanishes uniformly since the heat semigroup is contractive.

The tightness of the two terms about $Y^\perp$ and $Z^\perp$ follows from the following estimates. For all $p > 1$, there exists a finite constant C_p only depending on $\{V_x\}$ such that for all t_1, $t_2 \in [0,\infty)$ and $x_1, x_2 \in [0,1]$,

$$E\left|Y^\perp(t_1,x_1) - Y^\perp(t_2,x_2)\right|^{2p} \leq C_p(|t_1-t_2|^p + |x_1-x_2|^p); \tag{2.8}$$

$$E\left|Z^\perp(t_1,x_1) - Z^\perp(t_2,x_2)\right|^{2p} \leq C_p(|t_1-t_2|^{\frac{p}{2}} + |x_1-x_2|^p). \tag{2.9}$$

(2.8) and (2.9) are standard estimates for stochastic heat equations and the proof only involves computations, so we omit them here. □

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